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Application of Random Theory in Determining the Optimum Producing Well Numbers for Oil and Gas Reservoirs

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Abstract—Either in a horizontal direction or in a vertical direction, numerous physical measurements of the reservoirs, such as porosity, permeability, and saturation, are not evenly distributed, but are either restricted by general geologic rule or at random. So, the production rate of an individual well for oil and gas fields is stochastic. With the condition that the production rate is random, the authors of this paper try to apply random theory to the determination of optimum production well number so as to derive the calculating formula from examples. The result shows that the method is successful, and the corresponding calculation coincides with the practical environment.

Keywords—Oil reservoirs, Oil well, Probability, Γ distribution, Approximate calculation.

NOMENCLATURE

$E(\bullet)$	expectation of $(\bullet)$	K	formation effective permeability, μm^2
P_e	formation pressure, MPa	h	net reservoir thickness, m
P_{wf}	bottom hole flowing pressure, Mpa	r_e	boundary distance of gas-well delivery, m
Q_g	surface gas rate, 10^4 m^3/d	r_w	radius of well, m
μ_g	viscosity of gas	s_a	apparent skin factor
Z	gas deviation factor	r_g	relative density of natural gas
T	formation absolute temperature, K	β	inertia resistance coefficient
T_{sc}	surface standard temperature, K	B, c	parameters in Γ distribution
P_{sc}	surface standard pressure, Mpa		

1. INTRODUCTION

For nonhomogeneous oil and gas reservoirs, success in drilling is stochastic. Even if it is successful, the corresponding production rate is also a random variable. To satisfy the consumer's expected demand for total production rate for oil and gas reservoirs, it is necessary to drill enough

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wells. But drilling excessive wells would increase the cost and cause waste. So, we must plan the proper number of wells to be drilled. This is a key problem in designing the exploratory scheme for oil and gas fields.

2. ANALYSIS AND COMPUTATION

Let p be the probability of success when an individual well is to be drilled and $q = 1 - p$ for failure. Supposing N wells are drilled in a nonhomogeneous oil and gas deposit, the random variable η of successful well numbers follows the binomial distribution

$$P_k \equiv P(\eta = k) = \binom{N}{k} p^k q^{N-k}, \qquad k = 0, 1, 2, 3, \dots, N. \tag{1}$$

We define its generating function by $\varphi_\eta(s)$,

$$E(s^\eta) = \varphi_\eta(s) = \sum_{k=0}^{N} P_k s^k = (q + ps)^N. \tag{2}$$

Based on the study [1,2] for the permeability of the nonhomogeneous oil and gas deposits, and by means of Darcy's law, it can be seen that the successful individual well production rate is a random variable ξ that follows the Γ distribution and has density function

$$f_\xi(x) = \begin{cases} \dfrac{c^{B+1} x^B e^{-cx}}{\Gamma(B+1)}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{3}$$

where B and c are positive constants, and $\Gamma(B+1)$ is the Γ function

$$\Gamma(B+1) = \int_0^\infty x^B e^{-x}\, dx. \tag{4}$$

The characteristic function of ξ is

$$E\left(e^{it\xi}\right) = g_\xi(t) = \frac{1}{(1 - it/c)^{B+1}}. \tag{5}$$

When N wells are to be drilled, the total production rate Q is the sum of the production rates of the N individual wells (dry well production rate is 0),

$$Q = \sum_{j=1}^{\eta} \xi_j, \tag{6}$$

where ξ_j represents the j^{th} well production rate. Assuming that ξ_j are independent of each other, independent of η (see the Appendix), and follow the common Γ distribution, then the characteristic function of Q is

$$g_Q(t) = \varphi_\eta[g_\xi(t)]. \tag{7}$$

This is due to the following fact:

$$\begin{aligned} g_Q(t) = Ee^{itQ} = E\left[Ee^{itQ} \mid \eta\right] &= \sum_{k=0}^{N} E\left(e^{itQ} \mid \eta = k\right) \times P_k \\ = \sum_{k=0}^{N} E\left(e^{it\sum_{j=1}^{k}\xi_j}\right) \times P_k &= \sum_{k=0}^{N} [g_\xi(t)]^k P_k = \varphi_\eta[g_\xi(t)]. \end{aligned} \tag{8}$$

Thus, from (2) we get

$$\begin{aligned} g_Q(t) &= \varphi_\eta \left[\frac{1}{(1-it/c)^{B+1}} \right] = \sum_{k=0}^{N} P_k \left[\frac{1}{(1-it/c)^{B+1}} \right]^k \\ &= \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p}{(1-it/c)^{B+1}} \right)^k q^{N-k} = \left[\frac{p}{(1-it/c)^{B+1}} + q \right]^N . \end{aligned} \tag{9}$$

By means of the inverse Fourier transform, we may compute

$$\varphi(x) \equiv \int_{-\infty}^{\infty} e^{-itx} \left[\frac{p}{(1-it/c)^{B+1}} + q \right]^N dt. \tag{10}$$

Then, $(1/2\pi)\varphi(x)$ is the density function of Q.

Suppose the production rate demand for oil and gas deposit is $\tilde{Q}$. Then the probability that can satisfy the production rate demand for drilling N wells is

$$P\left(Q \geq \tilde{Q}\right) = \int_{\tilde{Q}}^{\infty} \frac{1}{2\pi}\varphi(x)\,dx = 1 - \int_{-\infty}^{\tilde{Q}} \frac{1}{2\pi}\varphi(x)\,dx. \tag{11}$$

Hence, for a given probability $1-\alpha$ (such as $\alpha = 0.01, 0.05$, and so on), the necessary well numbers to be drilled are

$$N^* = \min_{N \geq 1} \left\{ P\left(Q \geq \tilde{Q}\right) \geq 1 - \alpha \right\}. \tag{12}$$

That is to say, drilling N^* wells can bring the total production rate up to the expected demand $\tilde{Q}$ with probability $1-\alpha$, but the risk (the probability of failure to achieve the production rate demand) is only α.

3. THE METHOD FOR APPROXIMATE CALCULATION

The above method needs inverse Fourier transformation and program for numerous integration. For convenience, we give the following approximate algorithm.

Let the random variable ξ represent the arbitrary individual well production rate (including dry wells). As the individual well production rate for success follows the Γ distribution and the production rate for failure is zero, $E(\bar{\xi})$ represents the average production rate for an individual well to be drilled. (Note the difference between the $\bar{\xi}$ and individual well production rate ξ for success.) Then,

$$\begin{aligned} E\bar{\xi} &= E\left(\bar{\xi} \mid \text{success}\right) p + E\left(\bar{\xi} \mid \text{failure}\right) q \\ &= p \times \int_{-\infty}^{\infty} x f_\xi(x)\,dx + 0 \\ &= \frac{p}{\Gamma(B+1)} \int_0^{\infty} (cx)^{B+1} e^{-cx}\,dx \\ &= \frac{p}{c\Gamma(B+1)} \int_0^{\infty} y^{B+1} e^{-y}\,dy = \frac{p}{c\Gamma(B+1)}\Gamma(B+2) \\ &= \frac{p}{c}(B+1). \end{aligned} \tag{13}$$

In order to get the variance of $\bar{\xi}$, we first find the second moment of $\bar{\xi}$,

$$\begin{aligned} E\left(\bar{\xi}^2\right) &= p \times \int_0^{\infty} x^2 \frac{c^{B+1}}{\Gamma(B_1)} x^B e^{-cx}\,dx \\ &= \frac{p}{c\Gamma(B+1)} \int_0^{\infty} (cx)^{B+2} e^{-cx}\,dx \\ &= \frac{p\Gamma(B+3)}{c^2\Gamma(B+1)} = \frac{p}{c^2}(B+2)(B+1). \end{aligned} \tag{14}$$

So, the variance is

$$
\begin{aligned}
\sigma^2 &= \frac{p}{c^2}(B+2)(B+1) - \frac{p^2}{c^2}(B+1)^2 \\
&= \frac{p}{c^2}(B+1)(Bq+q+1) \\
&= \frac{p}{c^2}(B+1)\left[(B+1)q+1\right].
\end{aligned} \tag{15}
$$

The well numbers to be drilled for an oil and gas deposit are usually dozens and even hundreds; that is, N is large. By application of the central limit theorem to the condition of independent and identical distribution, we obtain

$$
\begin{aligned}
P\left(Q \le \tilde{Q}\right) &= P\left\{\frac{\sum\limits_{j=1}^{N} \xi_j - N \times E\bar{\xi}}{\sqrt{N}\sigma} \le \frac{\tilde{Q} - N \times E\bar{\xi}}{\sqrt{N}\sigma}\right\} \\
&\approx \int_{-\infty}^{(\tilde{Q}-N\times E\bar{\xi})/(\sqrt{N}\sigma)} \frac{1}{\sqrt{2\pi}} \bar{e}^{(t^2)/2}\, dt.
\end{aligned} \tag{16}
$$

So, the expected production rate $\tilde{Q}$ on level $1 - \alpha$ is given by

$$
P\left(Q \ge \tilde{Q}\right) \approx 1 - \int_{-\infty}^{M} \frac{1}{\sqrt{2\pi}} \bar{e}^{(t^2)/2}\, dt, \tag{17}
$$

where integral upper limit

$$
M = \frac{c\tilde{Q} - Np(B+1)}{\sqrt{Np(B+1)[(B+1)q+1]}}. \tag{18}
$$

When the probability $1-a$ is given, the drilling well number N^* that satisfies $P(Q \ge \tilde{Q}) \ge 1-\alpha$ has the following approximate value:

$$
N^* = \min_{N\ge 1}\left\{P\left(Q \ge \tilde{Q}\right) \ge 1 - \alpha\right\}. \tag{19}
$$

Obviously, N^* depends on the successful ratio P, parameters B and c in Γ distribution, expected production rate $\tilde{Q}$, and the risk a as well.

4. PRODUCTION RESULTS AND ANALYSIS

In a certain gas field, 26 exploration wells have been drilled, with 24 successes and two failures. Presented in Table 1 are the production rates with success (production rate $= 10^4\ \mathrm{m}^3/\mathrm{d}$).

Table 1. Production rates for each well.

No. (well)	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Production rate	1.297	3.54	5.42	4.301	5.506	4.5	4.415	10.9
No. (well)	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	Y_{16}
Production rate	1.58	16.08	15.8	2.386	3.537	1.1	0.707	1.922
No. (well)	Y_{17}	Y_{18}	Y_{19}	Y_{20}	Y_{21}	Y_{22}	Y_{23}	Y_{24}
Production rate	0.6785	2.846	5.16	0.363	6.512	0.941	3.843	1.067

Now, the prescribed annual production rate, according to the development scheme, is $10 \times 10^8\,\mathrm{m}^3/\mathrm{year}$. The average production per well, among 24 exploratory wells, is $4.350146 \times 10^4\,\mathrm{m}^3/\mathrm{d}$. If we use previous methods, then the needed production number is given by

$$n = \left(\frac{10 \times 10^8}{330 \times 4.650146 \times 10^4}\right) \times 1.4 \approx 98.$$

By application of the method proposed in this paper, we calculate the well numbers needed to be drilled in order to activate the expected annual production rate (plan production rate target). Statistical tests [1,2] show that the well production rate follows the Γ distribution. By estimating the production data (Table 1), the parameters in Γ distribution are given as follows:

$$B = 0.02262259, \qquad c = 0.235078.$$

Applying the central limit theorem presented in Section 3, we obtain the results in Table 2.

Table 2.

N^* / P / a	0.7	0.8	0.85	0.9	0.95	0.99
0.2	112	97	91	86	81	78
0.15	114	100	94	88	83	80
0.1	118	103	96	91	86	82
0.05	125	108	101	95	90	86
0.01	137	118	110	104	97	93

Figure 1 is a relationship between the well numbers and risk a with the different success ratios P for drilling. Figure 2 is a relationship between the well numbers and success ratio P with the different risk values a.

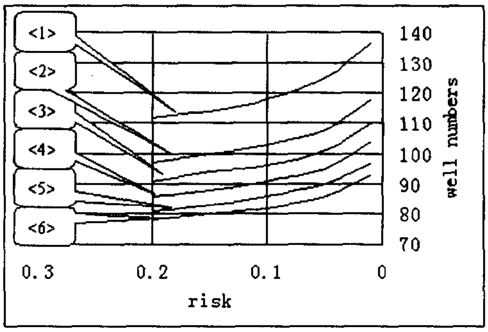

Figure 1. ⟨1⟩ $p = 0.7$, ⟨2⟩ $p = 0.8$, ⟨3⟩ $p = 0.85$, ⟨4⟩ $p = 0.9$, ⟨5⟩ $p = 0.95$, ⟨6⟩ $p = 0.99$.

Figure 1 shows that with a certain drilling success ratio, the well numbers will decrease as the drilling success ratio increases. In this example, the drilling success ratio is 0.9. To satisfy the plan target, the needed well numbers are shown in Table 3.

Table 3.

Risk a	0.2	0.15	0.1	0.05	0.01
Needed well numbers N^*	91	94	96	101	110

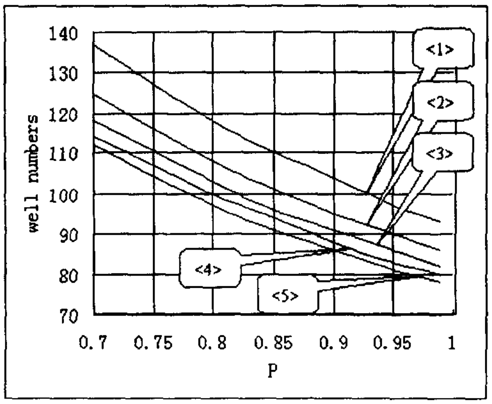

Figure 2. ⟨1⟩ $\alpha = 0.01$, ⟨2⟩ $\alpha = 0.05$, ⟨3⟩ $\alpha = 0.1$, ⟨4⟩ $\alpha = 0.15$, ⟨5⟩ $\alpha = 0.2$.

The above result coincides with the actual situation of the gas field.

5. CONCLUSIONS

From the results of this study, we obtain the following conclusions.

(1) The method presented in this paper, the application of probability theory to the determination of the optimum producing well numbers for oil and gas deposits, is feasible.
(2) The calculations in this example show that the result from the method coincides with field data. The calculating method is simple.
(3) This paper provides a new method by which we can determine producing well numbers as we design an exploitation plan for oil and gas fields, and reduce the blindness in determining numbers of wells to be drilled.

APPENDIX

THE INDEPENDENCE OF WELL PRODUCTION RATE

Based on the mechanics of porous flow, the production rate for a gas well can be obtained by the following equation:

$$P_e^2 - P_{wf}^2 = AQ_g + BQ_g^2,$$

where

$$A = \frac{8484\mu_g ZTP_{sc}}{KhT_{sc}}\left(\log\frac{r_e}{r_w} + 0.434 s_a\right),$$

$$B = \frac{1.966 \times 10^{-2} ZTr_g \beta P_{sc}^2}{Kh^2 T_{sc}^2}\left(\frac{1}{r_w} - \frac{1}{r_e}\right).$$

While the production pressure difference is given, the production rate can be represented by

$$Q_g = \frac{-A + \sqrt{A^2 + 4B\left(P_e^2 - P_{wf}^2\right)}}{2B}.$$

The computation, which is based on the production data, indicates that while other parameters are given, the production rate can be varied with well spacing, which is shown in Table 4. Figure 3 presents a relationship curve between the well spacing and the production rate. As can be seen from Figure 3, the well production rate will decrease gradually when the well spacing increases. But the range of decreasing is very small. In the calculation of this paper, since the change of

Table 4.

r_e (m)	500	1000	2000	3000	4000	5000	6000	8000	10000
Q_g 10^4 m^3/d	4.3456	4.169	4.0051	3.9142	3.852	3.8051	3.7675	3.709	3.6655

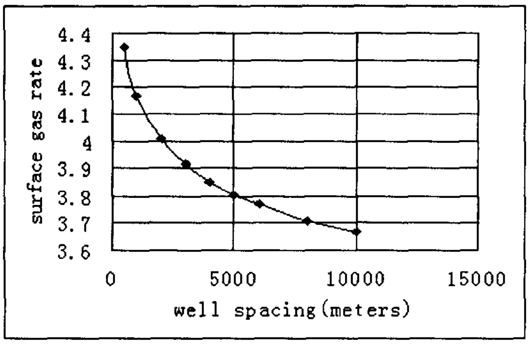

Figure 3. Surface gas rate (10^4 m^3/d).

the well spacing is not too large, we assume approximately that the well production rate is not related to well spacing; i.e., the well production rates are independent of each other.

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